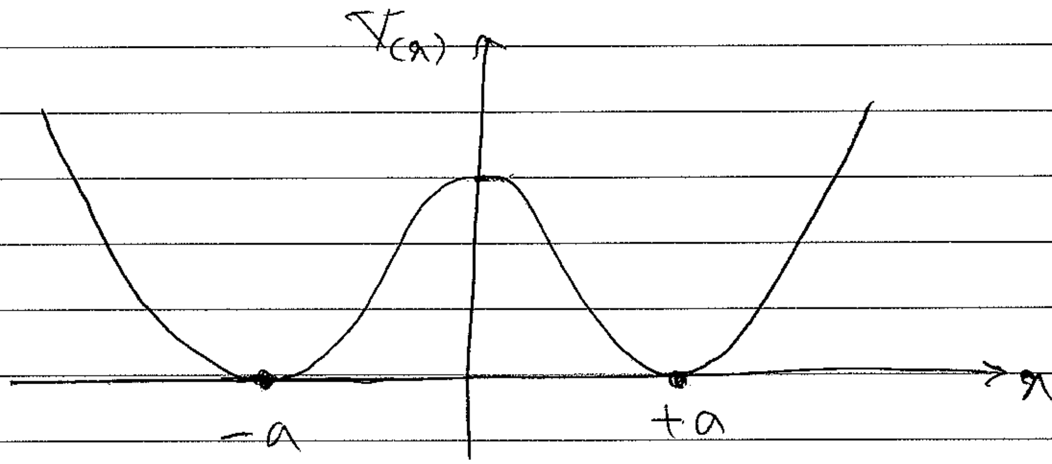
Lec 23: 10/21/2010

Path Integrals (Cont'd):

There are also cases where there is no classical trajectory. For example, consider the following potential:



Lets see whether the particle can start at $x = -a$ in the distant past $t \to -\infty$ and end up at $x = +a$ in the distant future $t \to +\infty$. In quantum mechanics there are infinite paths between these two points. However, there exists no classical trajectory connecting them. Starting at $x = -a$ when $t \to -\infty$

implies that the particle is at rest in the distant past. Then, being at the minimum of the potential, it will sit there forever classically. Thus the motion from $x=-a$, $t\to -\infty$ to $x=+a$, $t\to +\infty$ is purely a quantum mechanical phenomenon. The non-existence of a classical trajectory is conceptually fine. In practic, however, this leads to a complication.

We saw that if classical path(s) exist(s), then the main contribution to the path integral comes from paths that are sufficiently close to the classical trajectory.

Now, with no classical path, the question is how to perform the integration over the path. Where does the main contribution come from?

In such cases it is usually useful to use the imaginary time method.

To demonstrate the power of this method, lets consider $U(x,t;x')$ and calculate it two ways. We know that:

$$U(x,t;x') = \sum_n \psi_n^*(x')\psi_n(x)\, e^{-\frac{iE_n t}{\hbar}}$$

Going to imaginary time $\tau = it$, we have:

$$U(x,\tau;x') = \sum_n \psi_n(x')\psi_n(x)\, e^{-\frac{E_n\tau}{\hbar}}$$

In the limit $\tau \to \infty$, we find:

$$U(x,\tau;x') \to \psi_0(x')\psi_0(x)\, e^{-\frac{E_0\tau}{\hbar}}$$

Here E_0 and ψ_0 are the ground state energy and wavefunction respectively. Hence:

$$\lim_{\tau\to\infty} U(x,\tau;x) = |\psi_0(x)|^2 e^{-\frac{E_0\tau}{\hbar}}$$

On the other hand, in the path integral

formulation we have:

$$U(x,t;x') = A \sum_{\text{all paths}} e^{iS/\hbar}$$

$$S = \int_0^t \left[\frac{1}{2} m \dot{x}^2 - V_{(x)}\right] dt'$$

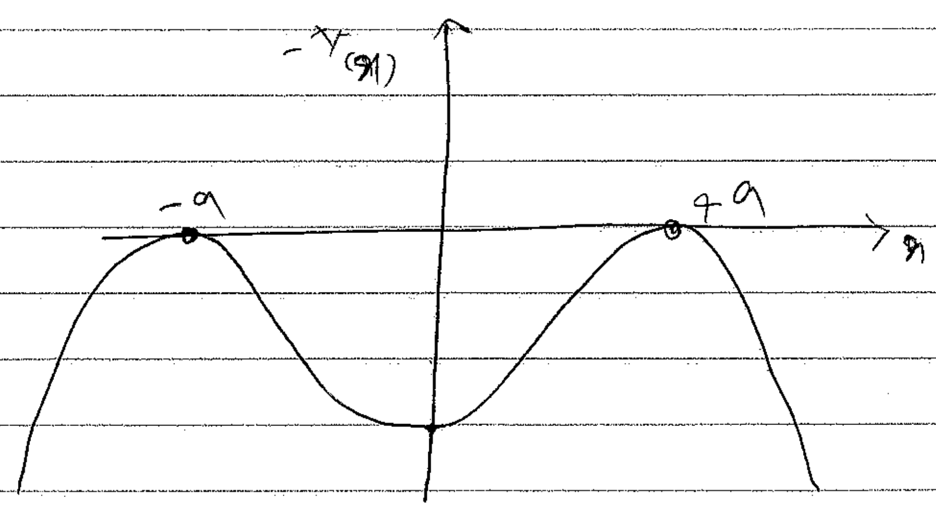
In the imaginary time $\frac{dx}{dt} = \frac{-i\,dx}{d\tau}$, and;

$$\frac{1}{2} m \left(\frac{dx}{dt}\right)^2 = -\frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2$$

This leads to:

$$S = -\int_0^\tau \left[\frac{1}{2} m \left(\frac{dx}{d\tau'}\right)^2 + V_{(x)}\right] d\tau$$

That amounts to flipping the potential. Now, for the double well potential we have;

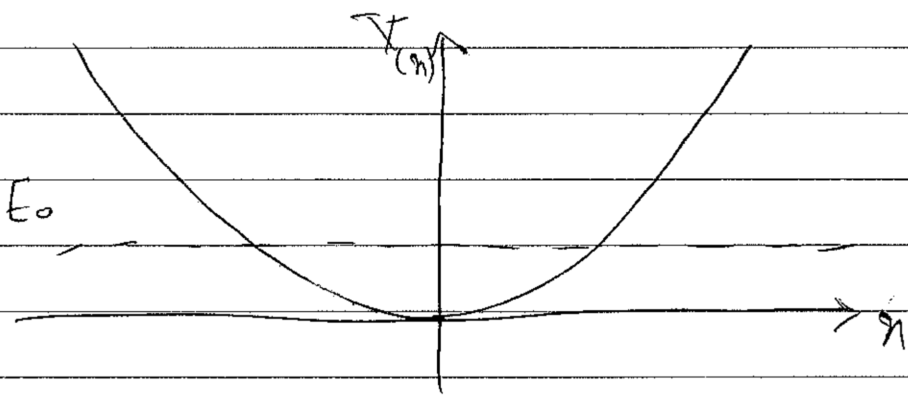


Note that we have a classical solution for going from $x = -a$ at $\tau \to -\infty$ to $x = +a$ at

$T \to +\infty$. This is very helpful since that path integral (after going to imaginary time) is dominated by the paths that are sufficiently close to the classical path mentioned above.

Similarly there is a classical trajectory that will take the particle from $x = -a$ at $T \to -\infty$ to $x = -a$ at $T \to +\infty$. Evaluating the path integral in the imaginary time, we can compare it with $|\Psi_0(-a)|^2 e^{-\frac{E_0 T}{\hbar}}$. Since the two are equal, we can then read $|\Psi_0(-a)|^2$ and E_0.

The imaginary time method can also be used to calculate the decay rate of a metastable state. For example, consider the following potential:

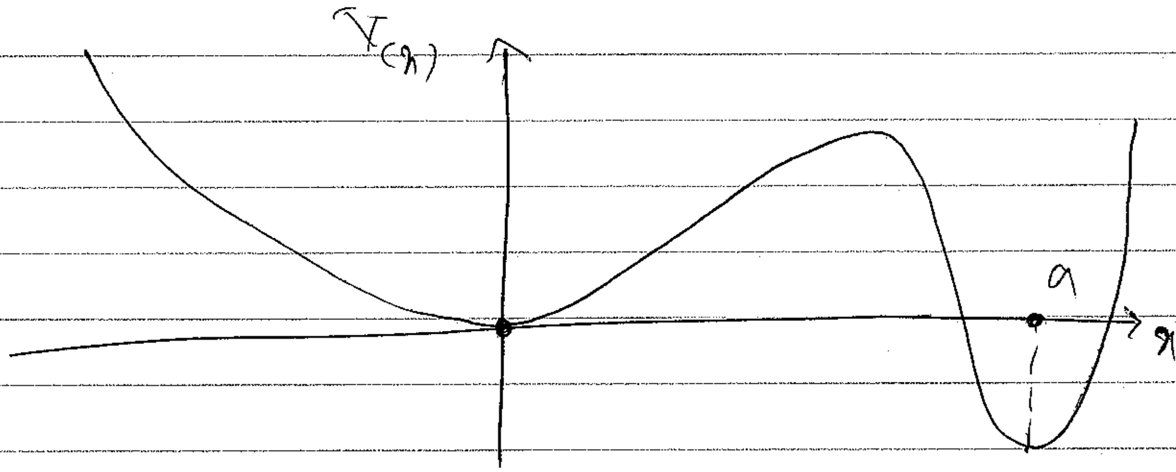


In the limit $\tau \to \infty$ (τ being the imaginary time) we have:

$$U(0, \tau; 0) \to |\Psi_0(0)|^2 e^{-\frac{E_0 \tau}{\hbar}}$$

$U(0, \tau, 0)$ can also be calculated via the path integral formulation. Note that the ground state wavefunction is peaked around $x=0$, and it is stable bound state.

Now lets assume that we change the shape of the potential in such a way that $V_{(x)}$ has two minima, one at $x=0$ and a second deeper one at $x=a$:

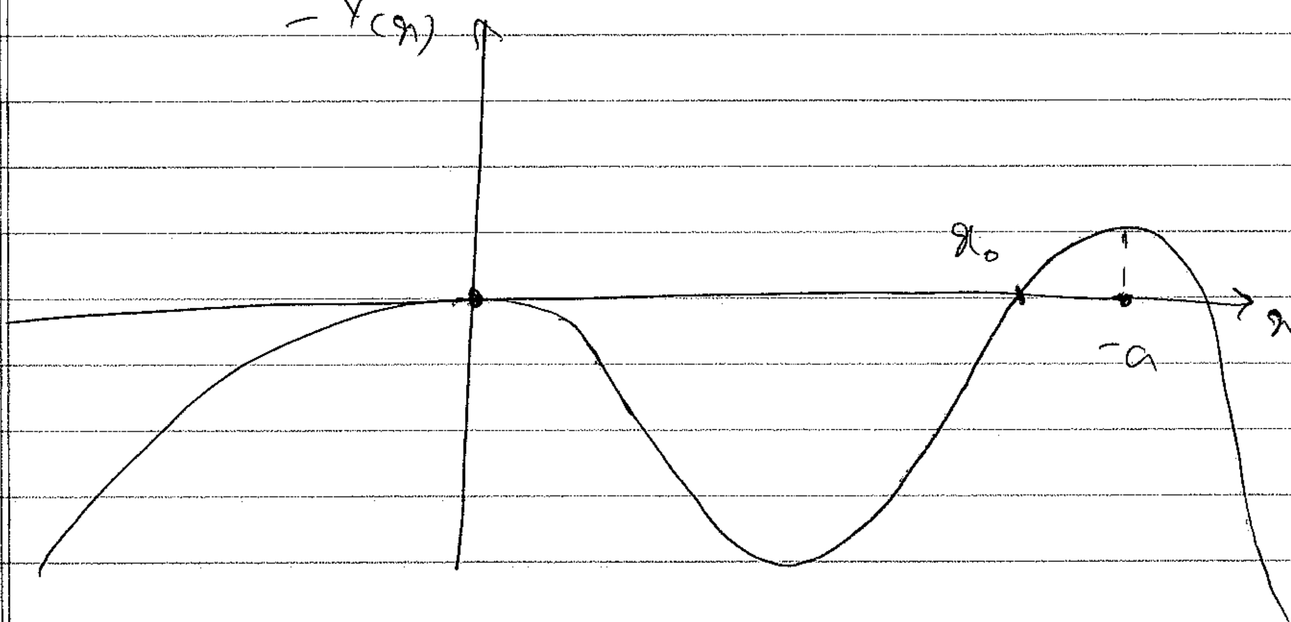


In this case the wavefunction $\psi_0(x)$, which is the ground state of the initial potential, will no longer be an energy eigenstate. Therefore it will evolve in time in a nontrivial way. Intuitively, we expect it to leak toward the true minimum at $x = a$. Hence E_0 must now have an imaginary component too:

$$U(0,\tau;0) \xrightarrow[\tau\to\infty]{} |\psi_0(0)|^2 e^{-\frac{(E_0+i\Gamma)\tau}{\hbar}}$$

The imaginary part Γ is the decay rate, or the tunneling rate, since in the real time it gives rise to a decaying exponential factor $e^{-\frac{\Gamma t}{2}}$ for the probability to find the particle at $x=0$.

Again we can calculate $U(0,T;0)$ in the path integral formalism. As we discussed, the potential is flipped when we go to the imaginary time.



It is seen that there exists a classical solution starting at $x=0$ at $T \to -\infty$ and ending at $x=0$ at $T \to +\infty$. This is the so called "Bounce solution" where the particle begins at $x=0$ when $\tau \to -\infty$, it rolls down and reaches x_0 at $\tau=0$, and then reverses the direction of its motion and settles at $x=0$ when $\tau \to +\infty$. Paths that are sufficiently close to this calssical trajectory make the main contribution to the path integral (in imaginary time).

Hence, finding $U(0,\tau;0)$ using path integrals, we can read Γ and obtain the tunneling rate.

We will discuss this more quantitatively after introducing the WKB method.